Explicit Numerical Approximations for SDDEs in Finite and Infinite Horizons using the Adaptive EM Method

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I. Backgruond

II. Convergence in finite time

III. Almost sure stability

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $w(t) = (w_t^1, \cdots, w_t^d)^T,\ t \geq 0$, be a d-dimensional Brownian motion defined on the probability space.
Consider the m-dimensional

$$dy(t) = f(y(t))dt + g(y(t))dw(t), \quad 0 \leq t \leq T, \tag{1}$$

with initial data $y(0) = y_0 \in \mathbb{R}^m$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^{m\times d}$.

Define the Euler-Maruyama (EM) approximate solution for the SDE (1). Given a stepsize $\Delta > 0$, let $t_k = k\Delta$ for $k \geq 0$. Compute the discrete approximations $X_k \approx y(t_k)$ by setting $X_0 = y_0$ and forming

$$X_{k+1} = X_k + f(X_k)\Delta + g(X_k)\Delta w_k, \tag{2}$$

where $\Delta w_k = w(t_{k+1}) - w(t_k)$. Let

$$\bar{X}(t) = X_k \quad \text{for } t \in [t_k, t_{k+1}) \tag{3}$$

and define the continuous EM approximate solution by

$$X(t) = X_0 + \int_0^t f(\bar{X}(s))ds + \int_0^t g(\bar{X}(s))dw(s). \tag{4}$$

Note that $X(t_k) = \bar{X}(t_k) = X_k$, that is $X(t)$ and $\bar{X}(t)$ coincide with the discrete approximate solution at the gridpoints.

Theorem

Under the global Lipschitz condition (GL),

$$E\left[\sup_{0\le t\le T}|X(t)-y(t)|^2\right]\le C\Delta, \tag{5}$$

where C is a positive constant dependent only on T, L, x_0 *but independent of* Δ*.*

(GL) There is a constant $L > 0$ such that

$$|f(x)-f(y)|\vee|g(x)-g(y)|\le L|x-y|$$

for all $x, y \in \mathbb{R}^m$.

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Theorem

Under the local Lipschitz condition (LL) and the linear growth condition (LG), the EM approximate solution converges to the exact solution of the SDE in the sense that

$$\lim_{\Delta \to 0} E\left[\sup_{0 \le t \le T} |X(t) - y(t)|^2\right] = 0.$$

Assumption (H0): For each $R = 1, 2, \cdots$, there is a pair of positive constants $L_R^{(1)}$ and $L_R^{(2)}$ such that

$$|f(x) - f(y)| \leq L_R^{(1)}|x - y| \text{ and } |g(x) - g(y)| \leq L_R^{(2)}|x - y|$$

for those $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq R$.

Theorem

Under Assumptions (H0) and the linear growth condition, if $L_R^{(1)} \leq \alpha_1 logR, (L_R^{(2)})^2 \leq \alpha_2 logR$ for some positive constants α_1 and α_2, then the order of the convergence is half, that is

$$E\left[\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2\right] = O(\Delta).$$

Counter example

Applying the EM to the SDE

$$dy(t) = (y(t) - y^3(t))dt + 2y(t)dB(t).$$

gives

$$X_{k+1} = X_k(1 + \Delta - X_k^2\Delta + 2\Delta B_k).$$

Lemma

Given any initial value $X_0 \neq 0$ and any $\Delta > 0$,

$$\mathbb{P}\Big(\lim_{k\to\infty} |X_k| = \infty\Big) > 0.$$

- Higham, J., Mao, X., Yuan, C., Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations. SIAM J. Numer. Anal. 45 (2007), 592-609.
- Wu, F., Mao, X., Szpruch, L., Almost sure exponential stability of numerical solutions for stochastic delay differential equations, Numerische Mathematik, 115 (2010), 681-697.

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Backward Euler (BE) Method

Given a step size $\Delta > 0$, set $Z_0 = x_0$ and compute

$$Z_{k+1} = Z_k + f(Z_{k+1})\Delta + g(Z_k)\Delta B_k \tag{6}$$

for $k = 0, 1, 2, \cdots$.

The BE method is implicit as for every step given Z_k, equation (6) needs to be solved for Z_{k+1}. For this purpose, some conditions need to be imposed on f.
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There exists a constant $\alpha \in \mathbb{R}$ such that

$$(u-v)^T(f(u)-f(v)) \le \alpha|u-v|^2, \quad \forall u,\ v \in \mathbb{R}^n.$$

Moreover, for any $R \ge 0$, there exists a positive constant K_R such that

$$|f(u)-f(v)| \le K_R|u-v|,$$

for any $u,\ v \in \mathbb{R}^n$, $|u| \vee |v| \le R$.
There exist constants $h_1 \in \mathbb{R}$ and $h_2 > 0$ such that

$$|u-v|^2|g(u)-g(v)|^2 - 2|(u-v)^T(g(u)-g(v))|^2 \le h_1|u-v|^4,$$

and

$$|g(u,j)-g(v,j)|^2 \le h_2|u-v|^2,$$

for any $u,\ v \in \mathbb{R}^n$.

- Li, X., Ma, Q., Yang, H., Yuan, C., The numerical invariant measure of stochastic differential equations with Markovian switching. SIAM J. Numer. Anal. 56 (2018), 1435-1455.

Truncated EM Method

Let $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $\mu(u) \to \infty$ as $u \to \infty$ and $\sup_{|x| \le u}(|f(x)| \vee |g(x)|) \le \mu(u)$. Let h be a strictly decreasing function such that $\lim_{\Delta \to 0} h(\Delta) = \infty, \Delta^{1/4} h(\Delta) \le \hat{h}$(constant) and define

$$\pi_\Delta(x) = (|x| \wedge \mu^{-1}(h(\Delta))\frac{x}{|x|}$$

$$f_\Delta(x) = f(\pi_\Delta(x)), \quad g_\Delta(x) = g(\pi_\Delta(x)).$$

The truncated EM method:

$$X_\Delta(t_{k+1}) = X_\Delta(t_k) + f_\Delta(X_\Delta(t_k))\Delta + g_\Delta(X_\Delta(t_k))\Delta w_k.$$

- L. Hu, X. Li, X. Mao, Convergence rate and stability of the truncated Euler-Maruyama method for stochastic differential equations, J. Comp. Appl. Math., 337(2018), 274-289.

Tamed EM Method

Define

$$|f_h(x)| = \frac{f(x)}{1 + h^\alpha |f(x)|}, |g_h(x)| = \frac{g(x)}{1 + h^\alpha |g(x)|}, \alpha \in (0, \frac{1}{2}].$$

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$$X_h(t_{k+1}) = X_h(t_k) + f_h(X_\Delta(t_k))\Delta + g_h(X_\Delta(t_k))\Delta w_k.$$

- Hutzenthaler, M., Jentzen, A. and Kloeden, P.E.: Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients. Ann. Appl. Probab. 22, (2012), 1611-1641.

- Fang, W., Giles, M.B. Adaptive Euler-Maruyama method for SDEs with nonglobally Lipschitz drift. Ann. Appl. Probab. 30 (2020), no. 2, 526-560.

Consider the following SDDEs

$$dY_t = (-2Y_t - Y_t^3 + \frac{1}{2} Y_t \sin(Y_{t-1}))dt + \sqrt{2} Y_t \cos(Y_{t-1}) dW_t \quad (7)$$

with initial data $\xi \in C([-1, 0]; \mathbb{R}), \xi(0) = c \in \mathbb{R}/\{0\}$. Using the result of Wu, Mao and Szpruch, we can show that the exact solution of the SDDE (7) is almost sure exponentially stable, i.e.

$$\limsup_{t\to\infty} \frac{1}{t} \log |Y_t| \leq -\lambda \text{ a.s.,} \qquad \lambda > 0.$$

However, the discrete (standard) EM approximate solution

$$\begin{cases} X_k & = \xi(k\Delta) \quad k = -m, -m+1, ..., 0, \\ X_{k+1} & = X_k - X_k[(2 + X_k^2 - \frac{1}{2}X_k \sin(X_{k-1}))\Delta + \sqrt{2}\cos(X_{k-1})\Delta W_k], \end{cases} \tag{8}$$

where $\Delta = 1/m, m \in \mathbb{N}$, is not almost sure exponentially stable. This means that it does not exist a constant $\eta > 0$ and a $\Delta^* \in (0, 1)$ such that for all $\Delta \in (0, \Delta^*)$

$$\limsup_{k\to\infty} \frac{1}{k\Delta} \log |X_k| \leq -\eta \ \text{ a.s. } .$$

Consider an m-dimensional stochastic differential delay equation

$$dY_t = f(Y_t, Y_{t-\tau})dt + g(Y_t, Y_{t-\tau})dW_t \tag{9}$$

on $t \geq 0$, and the initial data satisfies the following condition: for any $p \geq 2$

$$\{Y(\theta) : -\tau \leq \theta \leq 0\} = \xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^m),$$

that is ξ is a $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^m)$-valued random variable such that $E||\xi||^p < \infty$.

We now define the adaptive EM method for SDDEs:
For the interval $[-\tau, 0]$ we consider a fixed time step $\Delta := \tau/M$ where M is some positive integer. We define the discrete time approximate solution as

$$\hat{X}_{t_n} := \xi(t_n), \quad t_n := n\Delta, \quad n = 0, -1, ..., -M. \tag{10}$$

For $t > 0$, the time step is determined by a function $h^\delta : \mathbb{R}^m \to \mathbb{R}^+$ with $\delta \in (0, 1)$. Now, set

$$h_n^\delta := h^\delta(\hat{X}_{t_n}), \quad t_{n+1} := t_n + h_n^\delta, \quad \hat{t}_n := \max\{t_k : t_k \leq t_n - \tau, k = -M, ..., n\} \tag{11}$$

and for every $\omega \in \Omega$, $N(\omega) := \inf\{n \in \mathbb{Z}^+ : t_n(\omega) \geq T\}$. Then, for $n = 0, 1, ...N(\omega)$, we define

$$\hat{X}_{t_{n+1}} := \hat{X}_{t_n} + f(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})h_n^\delta + g(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})\Delta W_n. \tag{12}$$

Note that the adaptive time steps $\{h_n^\delta\}$, the discretization times $\{t_n\}$, and the number of steps N are all random variables.

We now define the the continuous-time approximate solution. For every $t \in [0, T]$, let

$$\bar{X}_t := \hat{X}_{t_n} \quad \text{for} \quad t \in [t_n, t_{n+1}) \tag{13}$$

and define

$$\begin{aligned} X_t &:= \xi(t), \quad t \in [-\tau, 0]; \\ X_t &:= X_0 + \int_0^t f(\bar{X}_s, \bar{X}_{s-\tau}) ds + \int_0^t g(\bar{X}_s, \bar{X}_{s-\tau}) dW_s, \quad t > 0. \end{aligned} \tag{14}$$

Convergence of the approximate solution for finite time interval

Assumption (H1): The functions f and g satisfy the local Lipschitz condition: for every $R > 0$ there exists a positive constant C_R such that

$$|f(x,y)-f(\bar{x},\bar{y})|+||g(x,y)-g(\bar{x},\bar{y})|| \leq C_R(|x-\bar{x}|+|y-\bar{y}|) \quad (15)$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^m$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$. Furthermore, there exist two constants $\alpha, \beta \geq 0$ such that for all $x, y \in \mathbb{R}^m$, f satisfies the one-sided linear growth condition:

$$\langle x, f(x,y)\rangle \leq \alpha(|x|^2 + |y|^2) + \beta \quad (16)$$

and g satisfies the linear growth condition:

$$||g(x,y)||^2 \leq \alpha(|x|^2 + |y|^2) + \beta. \quad (17)$$

Assumption (H2): The time step function $h^\delta : \mathbb{R}^m \to \mathbb{R}^+,\ \delta \in (0,1)$, is continuous, strictly positive and bounded by δT, i.e.

$$0 < h^\delta(x) \le \delta T \quad \text{for all } x \in \mathbb{R}^m. \tag{18}$$

Furthermore, there exist constants $\alpha, \beta > 0$ such that for all $x, y \in \mathbb{R}^m$.

$$\langle x, f(x,y)\rangle + \frac{1}{2} h^\delta(x)|f(x,y)|^2 \le \alpha(|x|^2 + |y|^2) + \beta. \tag{19}$$

Definition

We say that the time horizon T is attainable if $\{t_n\}$ reaches T in a finite number of steps N, i.e. for almost all $\omega \in \Omega$, there exists a $N(\omega)$ such that $t_{N(\omega)} = \sum_{n=0}^{N(\omega)} h^\delta(X_{t_n}) \geq T$.

Theorem

If the SDE (9) satisfies Assumption (H1) and the function h^δ satisfies Assumption (H2), then T is attainable and for all $p > 0$ there exists a constant $C > 0$ dependent on T and p, but independent of h_n^δ, such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^p\right] \leq C. \tag{20}$$

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Strong convergence of the approximate solution to the exact solution

Theorem

If the SDDE 9 satisfies Assumption (H1) and the time step function h^δ satisfies Assumption (H2), then for all $p > 0$

$$\lim_{\delta \to 0} \mathbb{E}\left[\sup_{0 \le t \le T} |X_t - Y_t|^p\right] = 0.$$

Assumption (H3): There exists a constant $L > 0$ such that for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^m$, f satisfies the one-sided Lipschitz condition

$$2\langle x - \bar{x}, f(x, y) - f(\bar{x}, \bar{y})\rangle \le L(|x - \bar{x}|^2 + |y - \bar{y}|^2) \qquad (21)$$

and g satisfies the (global) Lipschitz condition

$$||g(x, y) - g(\bar{x}, \bar{y})||^2 \le L(|x - \bar{x}|^2 + |y - \bar{y}|^2). \qquad (22)$$

In addition f satisfies the polynomial growth Lipschitz condition: there exist constants $\gamma, \lambda, q > 0$ such that for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^m$

$$|f(x, y) - f(\bar{x}, \bar{y})| \le (\gamma(|x|^q + |y|^q + |\bar{x}|^q + |\bar{y}|^q) + \lambda)(|x - \bar{x}| + |y - \bar{y}|). \qquad (23)$$

Furthermore, for any $s, t \in [-\tau, 0]$ and $q > 0$, there exists a positive constant Λ such that

$$\mathbb{E}||\xi(t) - \xi(s)|| \le \Lambda|t - s|^q. \qquad (24)$$

Theorem

If the SDE (9) *satisfies Assumption (H3) and the time-step function h satisfies Assumption H2, then for all* $p > 0$, *there exists a positive constant C independent of* δ *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t - Y_t|^p\right] \le C\delta^{p/2}.$$

Convergence of the approximate solution for infinite time interval

Assumption (H4): The functions f and g satisfy the local Lipschitz condition: for every $R > 0$ there exists a positive constant C_R such that

$$|f(x,y)-f(\bar{x},\bar{y})|+||g(x,y)-g(\bar{x},\bar{y})|| \le C_R(|x-\bar{x}|+|y-\bar{y}|) \quad (25)$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^m$ with $|x|, |y|, |\bar{x}|, |\bar{y}| \le R$. Furthermore, there exists constants $\alpha_1 > \alpha_2 \ge 0$ and $\beta > 0$, such that for all $x, y \in \mathbb{R}^m$, f satisfies the dissipative one-sided linear growth condition:

$$\langle x, f(x,y)\rangle \le -\alpha_1|x|^2 + \alpha_2|y|^2 + \beta, \quad (26)$$

and g is globally bounded:

$$||g(x,y)||^2 \le \beta. \quad (27)$$

Assumption (H5): For every δ, the time step function $h^\delta : \mathbb{R}^m \to \mathbb{R}^+$, is continuous and uniformly bounded by h^δ_{max}, where $h^\delta_{max} \in (0, \infty)$.
Furthermore, there exist constants $\alpha_1 > \alpha_2 \geq 0$ and $\beta > 0$, such that for all $x, y \in \mathbb{R}^m$.

$$\langle x, f(x,y) \rangle + \frac{1}{2} h^\delta(x) |f(x,y)|^2 \leq -\alpha_1 |x|^2 + \alpha_2 |y|^2 + \beta. \quad (28)$$

Lemma

If the SDDE (9) *satisfies Assumption (H4), then there exists a positive constant C such that for all $t \geq 0$*

$$\mathbb{E}\left[|Y_t|^p\right] \leq C. \tag{29}$$

Theorem

If the SDE (9) *satisfies Assumption (H4) and the function h^δ satisfies Assumption (H5), then for all $p > 0$ there exists a constant C dependent on $h_{max}, \beta, \alpha_1, \alpha_2$ and p, but independent of δ and t, such that for all $t \geq 0$,*

$$\mathbb{E}\left[|X_t|^p\right] \leq C. \tag{30}$$

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If the SDE (9) *satisfies Assumption (H4) and the function* h^δ *satisfies Assumption (H5), then for all* $p > 0$ *there exists a constant C dependent on* $h_{max}, \beta, \alpha_1, \alpha_2$ *and* p, *but independent of* δ *and* t, *such that for all* $t \geq 0$,

$$\mathbb{E}\left[|X_t|^p\right] \leq C. \quad (30)$$

Assumption H6: The functions f and g satisfy the local Lipschitz condition: for every $R > 0$ there exists a positive constant C_R such that

$$|f(x,y)-f(\bar{x},\bar{y})|+||g(x,y)-g(\bar{x},\bar{y})|| \leq C_R(|x-\bar{x}|+|y-\bar{y}|) \quad (31)$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^m$ with $|x|, |y|, |\bar{x}|, |\bar{y}| \leq R$. Furthermore, there exist constants α_1, α_2 and β satisfying

$$\alpha_1 > 2\alpha_2 \geq 0 \text{ and } \beta > 0, \quad (32)$$

such that for all $x, y \in \mathbb{R}^m$, f satisfies

$$\langle x, f(x,y)\rangle + \frac{1}{2}||g(x,y)||^2 \leq -\alpha_1|x|^2 + \alpha_2|y|^2. \quad (33)$$

Assumption H7: For every δ, the time step function $h^\delta : \mathbb{R} \to \mathbb{R}^+$, is continuous and there exist constants $\alpha_1 > \alpha_2 \geq 0$ and $\beta > 0$, such that for all $x, y \in \mathbb{R}^m$.

$$\langle x, f(x,y) \rangle + \frac{1}{2} h^\delta(x) |f(x,y)|^2 + \frac{1}{2} ||g(x,y)||^2 \leq -\alpha_1 |x|^2 + \alpha_2 |y|^2. \tag{34}$$

Furthermore, h^δ is uniformly bounded by the real number h^δ_{max}, where $h^\delta_{max} \in (0, \infty)$ is small enough such that

$$2\alpha_2 e^{\alpha_1 h_{max}} < \alpha_1. \tag{35}$$

Theorem (Exponential Stability)

Consider the SDDE (9) *with a one-dimensional Brownian motion. If f and g satisfy Assumption (H6) and h^{δ} satisfies Assumption (H7), then the adaptive approximate solution is almost sure exponentially stable, i.e. there exists a $\lambda > 0$ such that*

$$\limsup_{n\to\infty} \frac{\log|\hat{X}_{t_n}|}{t_n} \leq -\lambda \text{ a.s.}$$

Proof of Theorem

By the definition of the adaptive EM method, we have

$$
\begin{aligned}
|\hat{X}_{t_{n+1}}|^2 \le |\hat{X}_{t_n}|^2 &+ 2h_n(\langle \hat{X}_{t_n}, f(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})\rangle + \frac{1}{2}h_n|f(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})|^2 + \frac{1}{2}|g(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})|^2) \\
&+ 2\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})h_n, g(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})\Delta W_n\rangle + |g(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})|^2(|\Delta W_n|^2 - h_n)
\end{aligned}
$$

Using (H7), we obtain

$$
\begin{aligned}
|\hat{X}_{t_{n+1}}|^2 \le |\hat{X}_{t_n}|^2 &- 2\alpha_1 h_n|\hat{X}_{t_n}|^2 + 2\alpha_2 h_n|\hat{X}_{\hat{t}_n}|^2 + 2\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})h_n, g(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})\Delta W_n\rangle \\
&+ |g(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})|^2(|\Delta W_n|^2 - h_n).
\end{aligned}
$$

This implies

$$
\begin{aligned}
e^{\lambda t_{n+1}}|\hat{X}_{t_{n+1}}|^2 \le e^{\lambda t_n}|\hat{X}_{t_n}|^2 &+ 2\alpha_2 e^{\lambda t_{n+1}}|\hat{X}_{\hat{t}_n}|^2 h_n + e^{\lambda t_{n+1}}|g(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})|^2(|\Delta W_n|^2 - h_n) \\
&+ 2e^{\lambda t_{n+1}}\langle \hat{X}_{t_n} + f(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})h_n, g(\hat{X}_{t_n}, \hat{X}_{\hat{t}_n})\Delta W_n\rangle.
\end{aligned}
$$

This means

$$e^{\lambda t_n}|\hat{X}_{t_n}|^2 \le |X_0|^2 + C + \bar{C}\sum_{k=0}^{n-1} e^{\lambda t_k}|g(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})|^2(|\Delta W_k|^2 - h_k)$$
$$+ \hat{C}\sum_{k=0}^{n-1} e^{\lambda t_k}\langle \hat{X}_{t_k} + f(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})h_k, g(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})\Delta W_k\rangle$$
$$\le C + \bar{C}\{M_n + N_n\},$$

where:

- $M_n := \sum_{k=0}^{n-1} e^{\lambda t_k}|g(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})|^2(|\Delta W_k|^2 - h_k)$;
- $N_n := \sum_{k=0}^{n-1} e^{\lambda t_k}\langle \hat{X}_{t_k} + f(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})h_k, g(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})\Delta W_k\rangle$.

This means

$$
\begin{aligned}
e^{\lambda t_n}|\hat{X}_{t_n}|^2 \le |X_0|^2 + C + \bar{C}\sum_{k=0}^{n-1} e^{\lambda t_k}|g(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})|^2(|\Delta W_k|^2 - h_k) \\
+ \hat{C}\sum_{k=0}^{n-1} e^{\lambda t_k}\langle \hat{X}_{t_k} + f(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})h_k, g(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})\Delta W_k\rangle \\
\le C + \bar{C}\{M_n + N_n\},
\end{aligned}
$$

where:

- $M_n := \sum_{k=0}^{n-1} e^{\lambda t_k}|g(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})|^2(|\Delta W_k|^2 - h_k)$;
- $N_n := \sum_{k=0}^{n-1} e^{\lambda t_k}\langle \hat{X}_{t_k} + f(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})h_k, g(\hat{X}_{t_k}, \hat{X}_{\hat{t}_k})\Delta W_k\rangle$.

We can show that $M + N$ is a local martingale with respect to $\{\mathcal{F}_{t_n}\}$. Thus by the discrete semimartingale convergence theorem one can see that

$$\lim_{n\to\infty}(M_n + N_n) < \infty \ \text{ a.s.}$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{t_n} \log(e^{\lambda t_n}|\hat{X}_{t_n}|^2) \leq 0 \text{ a.s.}$$

This is

$$\limsup_{n\to\infty} \frac{\log|\hat{X}_{t_n}|}{t_n} \leq -\frac{\lambda}{2} \text{ a.s.}$$

The proof is therefore complete.

Simulations

In Figure 1, we graphed the logarithm of EM solution.

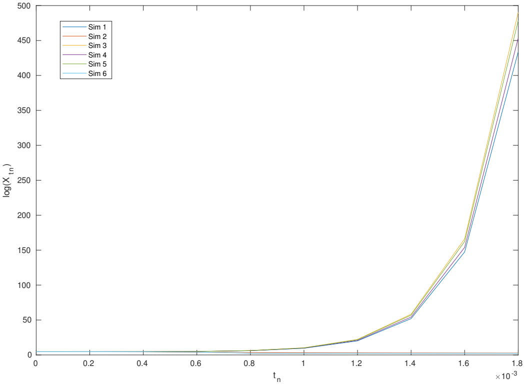

Figure: Simulations of the logarithm of the EM solution for $\Delta = 2 \times 10^{-3}$

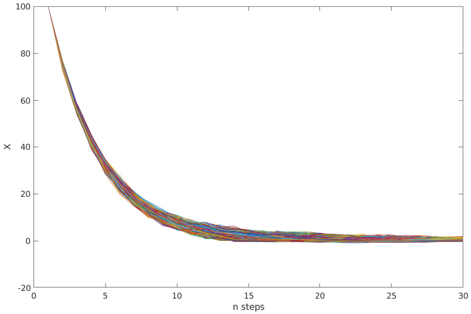

Figure: Simulations of adaptive-EM solution